Probability

Probability is a numerical measure of the chance (or likelihood) that a particular event will occur.

Probability values are always assigned on a scale of 0 to 1:

$$0 \le P(E) \le 1$$

A probability near 0 indicates that an event is unlikely to occur, whereas near 1 indicates that an event is likely to occur.

Total sum of probabilities of n simple events is equal to 1:

$$\sum_{k=1}^{n} P(E_k) = P(E_1) + P(E_2) + \cdots + P(E_n) = 1$$

Two approaches to probability

- **Classical**
- **Empirical (Frequency)**

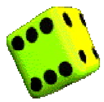
Classical probability

$$P(E) = \frac{n(E)}{n(S)}$$

Note: It is the same as the relative frequency of the event of equally likely events.

Examples:

1) Rolling a die: $S = \{1,2,3,4,5,6\}, E_1 = \{1\}, E_2 = \{2\}, \dots, E_6 = \{6\}$



$$P(1) = P(E_1) = \frac{n(E_1)}{n(S)} = \frac{1}{6} = 16.67\%$$

2) Santa Claus has 40 balls in his bag: 30 Red and 10 Blue. He randomly picks a ball to present to a boy. $S = \{r_1, \dots, r_{30}, b_1, \dots, b_{10}\}, R = \{r_1, \dots, r_{30}\}$.

$$P(R) = \frac{n(R)}{n(S)} = \frac{30}{40} = 75\%$$

Exercise: Find: 1) $\sum_{i=1}^{6} P(i)$; 2) $P(R) + P(B)$.

Empirical probability

Empirical probability of an event refers to the frequency of similar events happened in the past.

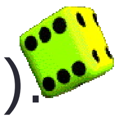
Example:

In 2006 after exceptionally heavy rainfall, parts of Britain experienced floods. For a total of 90 rainy days there were 2 floods. On the basis of this information, what is the probability that a future rainy day would cause floods?

$$P(Flood) = \frac{Floods}{Rainy\ days} = \frac{2}{90} = 0.022$$

Exercise

1) Tossing a die. S = {1, 2, 3, 4, 5, 6}. A = {2, 4, 6}, B = {1, 3, 5}, C = {2, 3, 5}. Find: 1) $P(A)$; 2) $P(A \cup B)$; 3) $P(A \cap C)$; 4) $P(A \cap B)$.



2) A radio station asked a random sample of 1000 out of the 250,000 listeners to find out how many preferred one of four types of music.

Type of music	Number of listeners
Popular	521
Big band	188
Classical	207
Other	84
Total	1000

What is the probability that a randomly chosen listener from the sample likes pop music?

The addition rule for union

The addition rule is used to compute the probability of the union of two events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example: Out of 200 students taking a mathematics course 160 passed the midterm exam, 140 passed the final exam and 124 students passed both exams. After reviewing the grades, the professor decides to give a passing grade to any student who passed at least one of the two exams. What is the probability of passing this course?

Let M = {pass midterm}, F = {pass final}, $M \cap F$= {pass both}, then:

$$P(M) = \frac{160}{200} = 0.80, P(F) = \frac{140}{200} = 0.70, P(M \cap F) = \frac{124}{200} = 0.62$$

According to the addition rule:

$$P(M \cup F) = P(M) + P(F) - P(M \cap F) = 0.80 + 0.70 - 0.62 = 0.88$$

Relationships of events

- Mutually exclusive events
- Dependent events
- Independent events

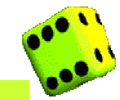
Mutually exclusive (disjoint) events

The events A and B are said to be mutually exclusive if the events do not have any experimental outcomes in common:

$$A \cap B = \emptyset \quad (P(A \cap B) = 0)$$

Examples: 1) f A = {a person is over 60 years old}, B = {a person is under 18}.

2) Rolling a die once and getting a 2 and a 3.



Addition rule (simplified) for mutually exclusive events:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B)$$

Example: For the rolling a die, let $A = \{2\}$, $B = \{3\}$, then:

$$P(A \cup B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Compliment rule:

$$P(A) = 1 - P(A') \quad and \quad P(A') = 1 - P(A) \quad and \quad P(A) + P(A') = 1$$

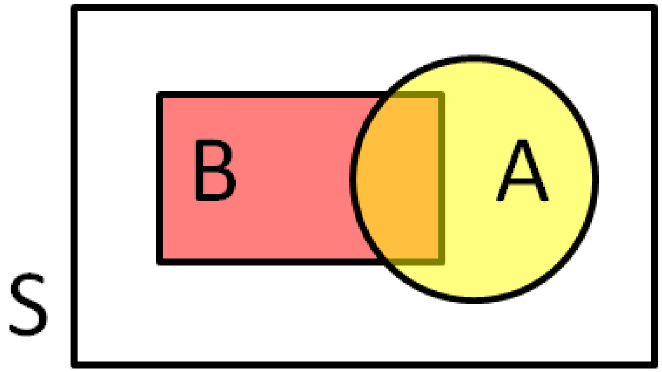
Example: For the rolling a die, let $A = \{2,3,4,5,6\}$, then:

$$P(A) = 1 - P(A') = 1 - \frac{1}{6} = \frac{5}{6}$$

Dependent/Independent events

If an event B has an influence on another event A, then the event A is called a dependent event (i.e. A depends on B).

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

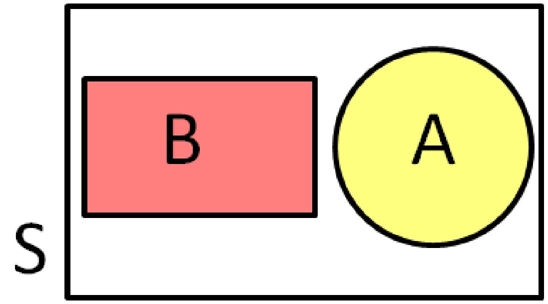


Example: Let A = {an accident}, B = {a driver has drunk}.

The events A and B are dependent, because the probability of an accident increases if the driver has drunk alcohol.

If an event B has no effect on another event A, then these events are called independent.

$$P(A|B) = P(A)$$



Example: Let A = {pass math exam}, B = {blue eyes}. The events are not influenced by each other, therefore, they are independent. $P(A|B) = P(A)$.

Multiplication rule for intersection

The multiplication rule is used to compute the probability of the intersection of two events

$$P(A \cap B) = P(B) \cdot P(A|B) \quad \left(derives\ from\ P(A|B) = \frac{P(A \cap B)}{P(B)} \right)$$

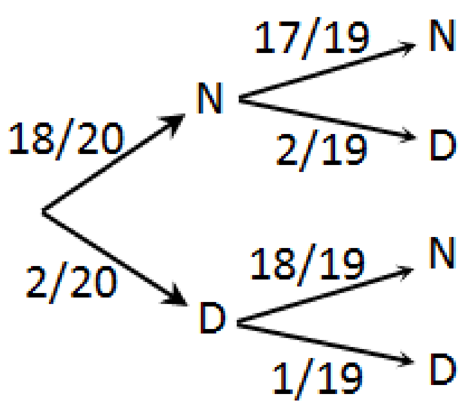
The multiplication rule (simplified) for independent events:

$$P(A \cap B) = P(B) \cdot P(A)$$

Probability tree diagram

Probability tree diagram is used to show the outcomes of an experiment and their relative probabilities.

Example: 2 out of 20 products are defective at a store. If a customer randomly buys two of the products, what is the probability of both being defective?



$$P(N \cap N) = P(N) \cdot P(N|N) = \frac{18}{20} \cdot \frac{17}{19} = 0.8053$$

$$P(N \cap D) = P(N) \cdot P(D|N) = \frac{18}{20} \cdot \frac{2}{19} = 0.0947$$

$$P(D \cap N) = P(D) \cdot P(N|D) = \frac{2}{20} \cdot \frac{18}{19} = 0.0947$$

$$P(D \cap D) = P(D) \cdot P(D|D) = \frac{2}{20} \cdot \frac{1}{19} = 0.0053$$

Exercise: What is the probability of at least one of them being defective?

Exercise

Let $P(A)$ = 0.6, $P(B)$ = 0.45 and $P(A \cap B)$ = 0.3.

a) Find $P(A \cup B)$. b) Find $P(A|B)$. c) Find $P(B|A)$.

d) Are events A and B independent?

e) Are events A and B mutually exclusive?

Solution:

a) ____

b) ______________________

c) ______________________

d) ____________________________________

e) ______________________

Joint probability table

Contingency table is a table of joint-occurrence frequencies of two variables.

Joint probability table is a table with probabilities calculated based on the contingency table.

Example: The table shows the numbers of full-time (F) and part-time (P) workers among 100 employees.

	F	**P**	**Totals**
Men	50	20	70
Women	25	5	30
Totals	75	25	100

$\Rightarrow$

	F	**P**	**Totals**
Men	0.5	0.2	0.7
Women	0.25	0.05	0.3
Totals	0.75	0.25	1

Find the probabilities that: **a)** an employee is a man and part-timer;

b) a man is part-timer; **c)** a woman is full-timer.

Bayes' Formula

Bayes' formula is used to calculate the posterior (updated, revised with an additional) probabilities:

Suppose $A_1, A_2, \ldots, A_n$ are pairwise disjoint events, whose union is the sample space. Then for an event E and for each i with $1 \leq i \leq n$:

$$P(A_i|E) = \frac{P(A_i) \cdot P(E|A_i)}{P(A_1) \cdot P(E|A_1) + \cdots + P(A_n) \cdot P(E|A_n)}$$

Example: A firm receives 65% of its parts from one supplier A_1 and 35% from a second supplier A_2. The table shows the percentages of good G and defective D parts received from the two suppliers.

	G	**D**
$\boldsymbol{A_1}$	98	2
$\boldsymbol{A_2}$	95	5

Given:

$P(A_1) = 0.65$, $P(A_2) = 0.35$,

$P(G|A_1) = 0.98$, $P(D|A_1) = 0.02$,

$P(G|A_2) = 0.95$, $P(D|A_2) = 0.05$.

Bayes' Formula (continued)

Suppose that a machine broke down due to a defective part and the firm wants to know the probability that the part came from A_1 and the probability that it came from A_2. That is: $P(A_1|D)$ and $P(A_2|D)$.

$$P(A_1|D) = \frac{P(A_1) \cdot P(D|A_1)}{P(A_1) \cdot P(D|A_1) + P(A_2) \cdot P(D|A_2)} = \frac{0.65 \cdot 0.02}{0.65 \cdot 0.02 + 0.35 \cdot 0.05} = 0.426$$

Exercise: Find $P(A_2|D)$.

Reading

1) Murray R. Spiegel, *Schaum's outline of Theory and Problems of Probability and Statistics*, McGraw-Hill, 23 edition, 1998.

2) Nitis Mukhopadhyay, Probability and Statistical Inference, Marcel Dekker, Inc. 2000.